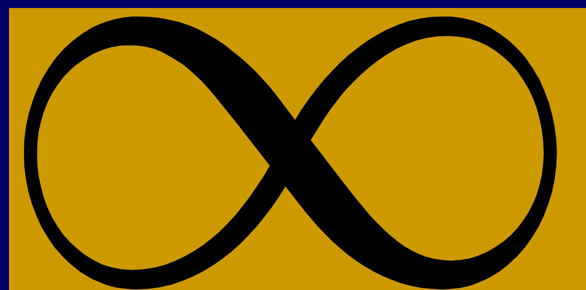
Great Theoretical Ideas In Computer Science

Flirting with Infinity



Lecture 14 Blake Scholl CS 15-251

The Ideal Computer: no bound on amount of memory

Whenever you run out of memory, the computer contacts the factory. A maintenance person is flown by helicopter and attaches 100 Gig of RAM and all programs resume their computations, as if they had never been interrupted.

An Ideal Computer Can Be Programmed To Print Out:

π: 3.14159265358979323846264…

2: 2.0000000000000000000000…

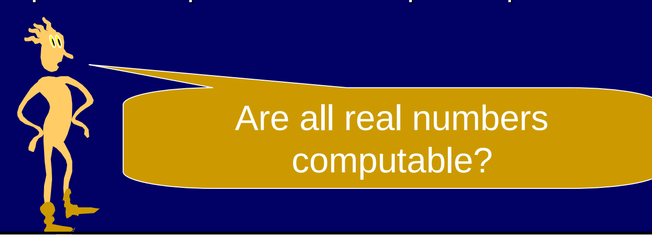
e: 2.7182818284559045235336…

1/3: 0.33333333333333333333….

ϕ: 1.6180339887498948482045…

Computable Real Numbers

A real number r is <u>computable</u> if there is a program that prints out the decimal representation of r from left to right. Thus, each digit of r will eventually be printed as part of the output sequence.



Describable Numbers

A real number r is <u>describable</u> if it can be unambiguously denoted by a finite piece of English text.

2: “Two.”

π: “The area of a circle of radius one.”

Theorem: Every computable real is also describable

Proof: Let r be a computable real that is output by a program P. The following is an unambiguous denotation:

“The real number output by:” P

MORAL: A computer program can be viewed as a description of its output.

Are all real numbers describable?

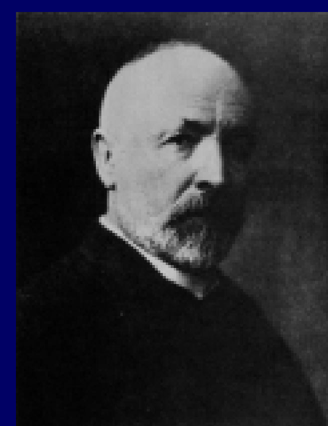
Correspondence Principle

If two finite sets can be placed into 1-1 onto correspondence, then they have the same size.

Correspondence Definition

Two finite sets are defined to have the the <u>same size</u> if and only if they can be placed into 1-1 onto correspondence.

Georg Cantor (1845-1918)



Cantor's Definition (1874)

Two sets are defined to have the same size if and only if they can be placed into 1-1 onto correspondence.

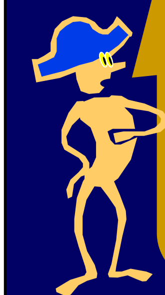
Cantor's Definition (1874)

Two sets are defined to have the same cardinality if and only if they can be placed into 1-1 onto correspondence.

Do ℕ and 𝔼 have the same cardinality?

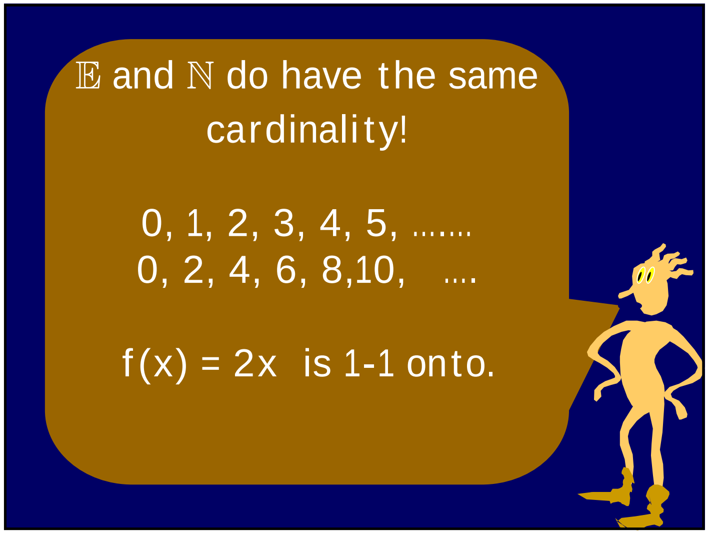
ℕ = { 0, 1, 2, 3, 4, 5, 6, 7, ... }

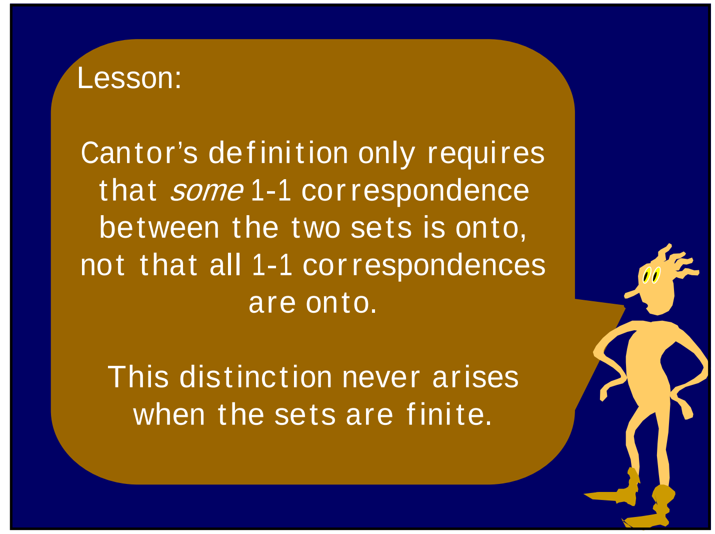
𝔼 = The even, natural numbers.

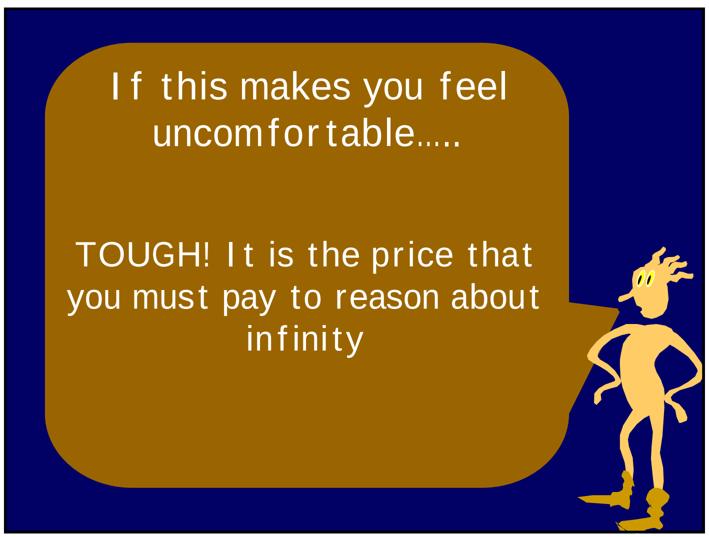


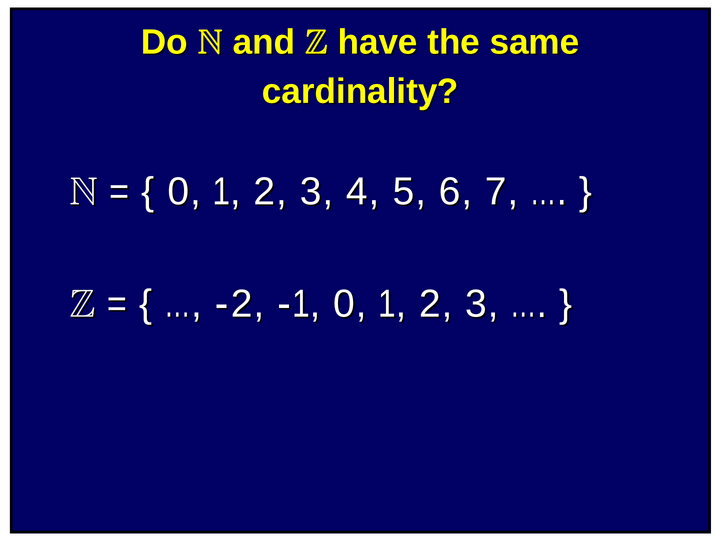
𝔼 and ℕ do not have the same cardinality! 𝔼 is a proper subset of ℕ with plenty left over.

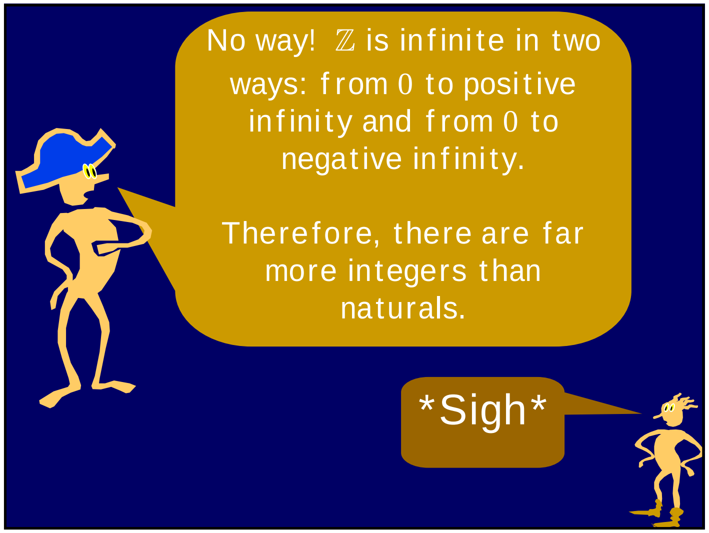
The attempted correspondence f(x)=x does not take 𝔼 *onto* ℕ.

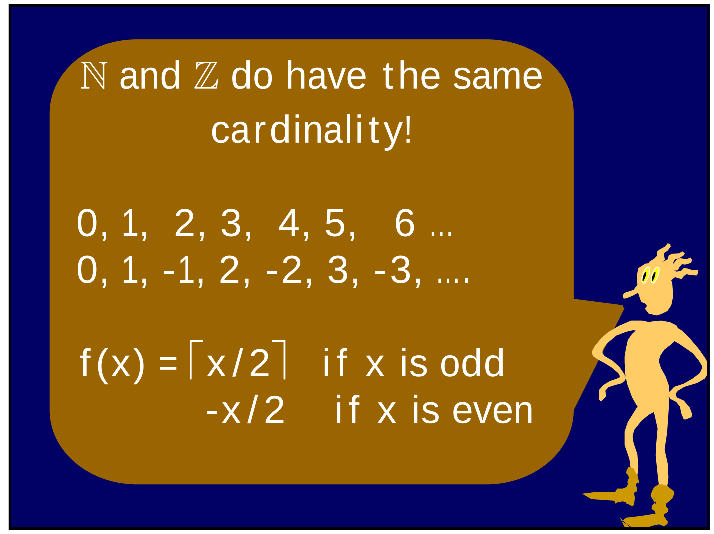

𝔼 and ℕ do have the same cardinality!
0, 1, 2, 3, 4, 5, ...
0, 2, 4, 6, 8,10, ...
f(x) = 2x is 1-1 onto.


Lesson:
Cantor's definition only requires that *some* 1-1 correspondence between the two sets is onto, not that all 1-1 correspondences are onto.
This distinction never arises when the sets are finite.


If this makes you feel uncomfortable.....
TOUGH! It is the price that you must pay to reason about infinity


Do ℕ and ℤ have the same cardinality?
ℕ = { 0, 1, 2, 3, 4, 5, 6, 7, ... }
ℤ = { ..., -2, -1, 0, 1, 2, 3, ... }





Transitivity Lemma

If f: A→B 1-1 onto, and g: B→C 1-1 onto
Then h(x) = g(f(x)) is 1-1 onto A→C

Hence, ℕ, 𝔼, and ℤ all have the same cardinality.

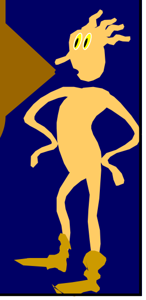
Do ℕ and ℕxℕ have the same cardinality?

ℕ= { 0, 1, 2, 3, 4, 5, 6, 7, …. }

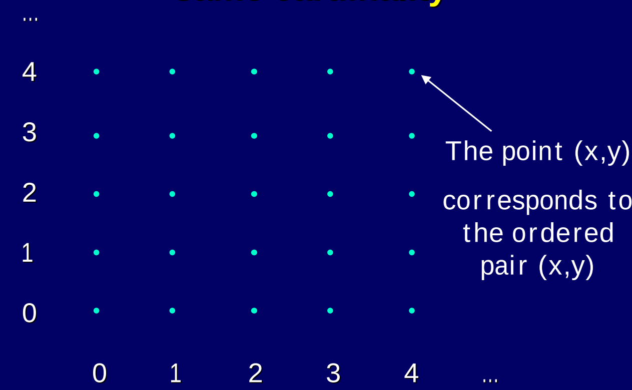
ℕxℕ = Pairs of natural numbers (x,y)

You bet!

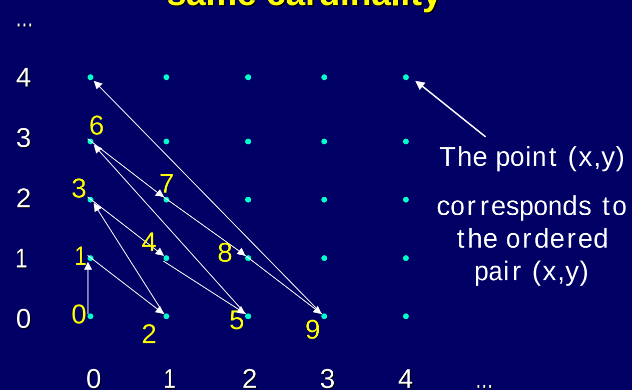
We can prove this graphically.



Theorem: ℕ and ℕxℕ have the same cardinality



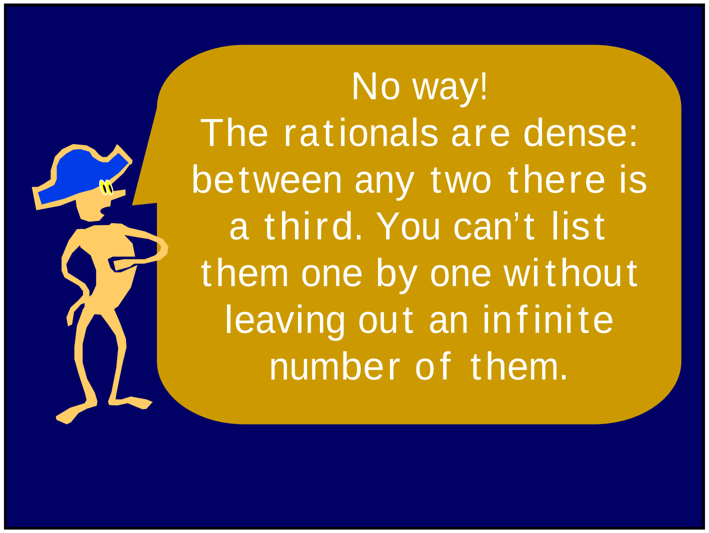
Theorem: ℕ and ℕxℕ have the same cardinality

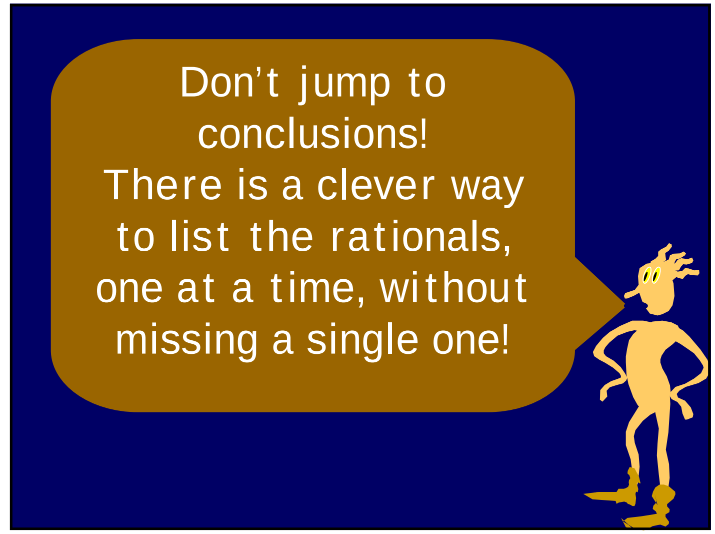


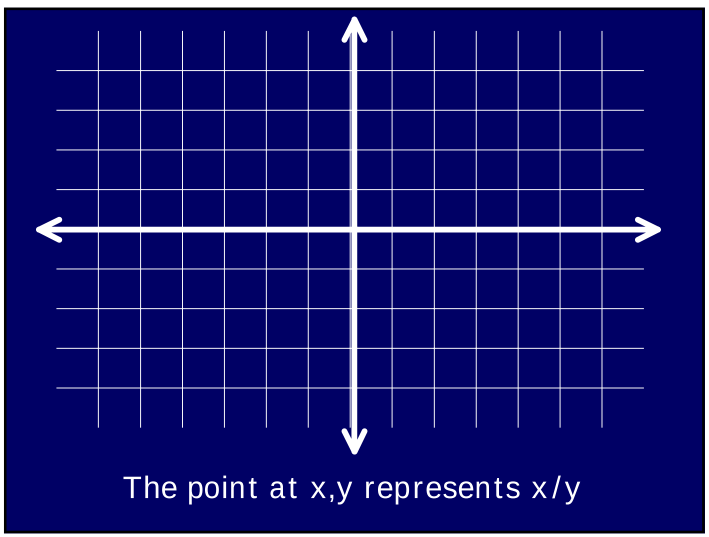
Do ℕ and ℚ have the same cardinality?

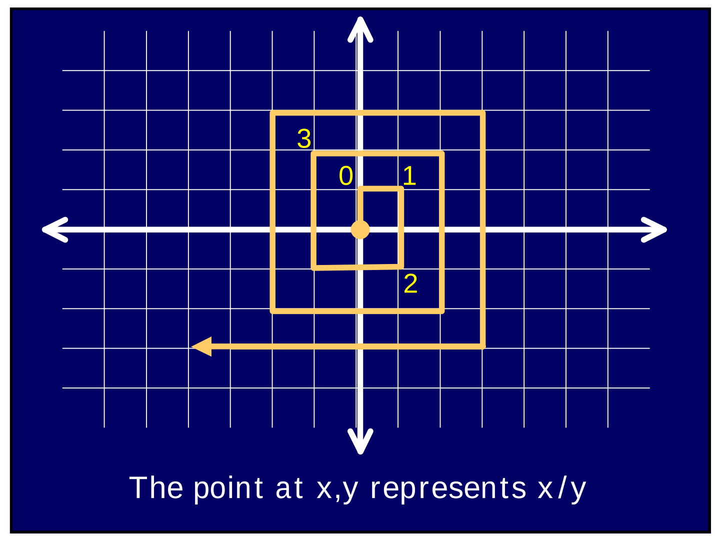
ℕ= { 0, 1, 2, 3, 4, 5, 6, 7, ... }

ℚ = The Rational Numbers


No way!
The rationals are dense: between any two there is a third. You can't list them one by one without leaving out an infinite number of them.

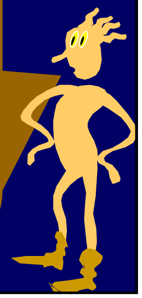

Don't jump to conclusions!
There is a clever way to list the rationals, one at a time, without missing a single one!


The point at x,y represents x/y


3
0
1
2
The point at x,y represents x/y

We call a set <u>countable</u> if it can be placed into 1-1 onto correspondence with the natural numbers.

So far we know that N, E, Z, and Q are countable.

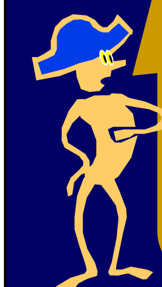


Do $\mathbb{N}$ and $\mathbb{R}$ have the same cardinality?

$\mathbb{N}$ = { 0, 1, 2, 3, 4, 5, 6, 7, …. }

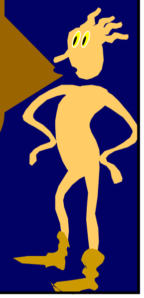
$\mathbb{R}$ = The Real Numbers

No way!
You will run out of natural numbers long before you match up every real.



Don't jump to conclusions! You can't be sure that there isn't some clever correspondence that you haven't thought of yet.

I am sure!
Cantor proved it.
He invented a very important technique called "DIAGONALIZATION".



Theorem: The set I of reals between 0 and 1 is not countable.

Proof by contradiction:

Suppose I is countable. Let f be the 1-1 onto function from ℕ to I. Make a list L as follows:

0: decimal expansion of f(0)
1: decimal expansion of f(1)

...

k: decimal expansion of f(k)

...

Theorem: The set I of reals between 0 and 1 is not countable.

Proof by contradiction:

Suppose I is countable. Let f be the 1-1 onto function from ℕ to I. Make a list L as follows:

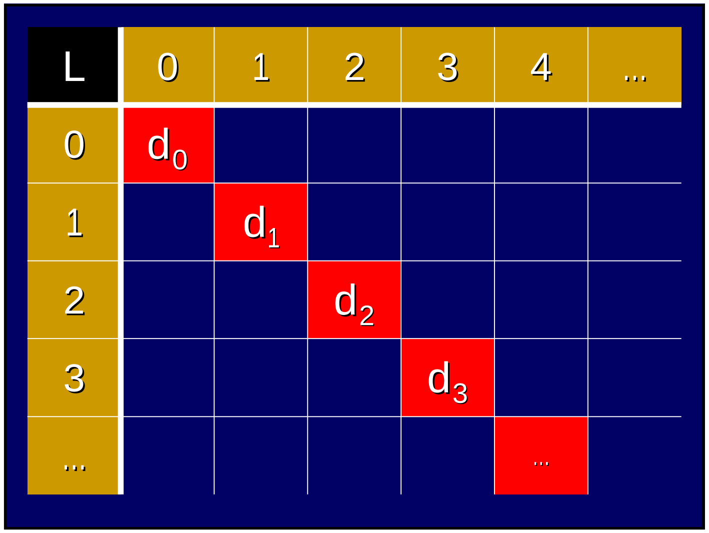
0: .33333333333333333333333...
1: .314159265657839593859 4982..

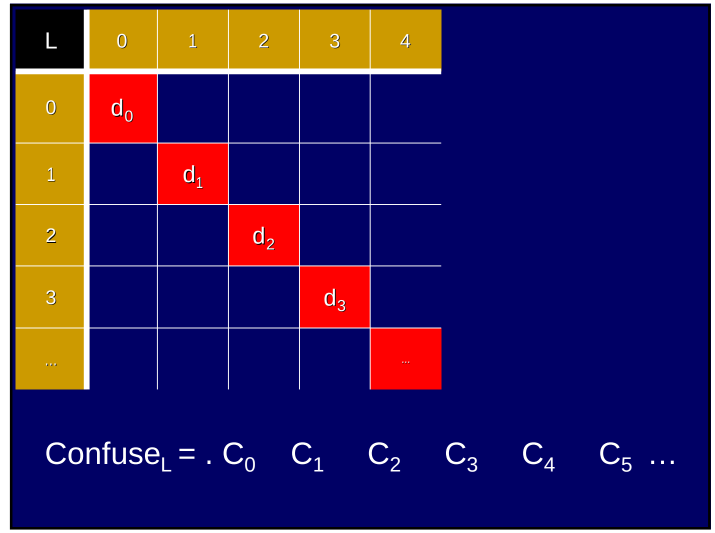
...

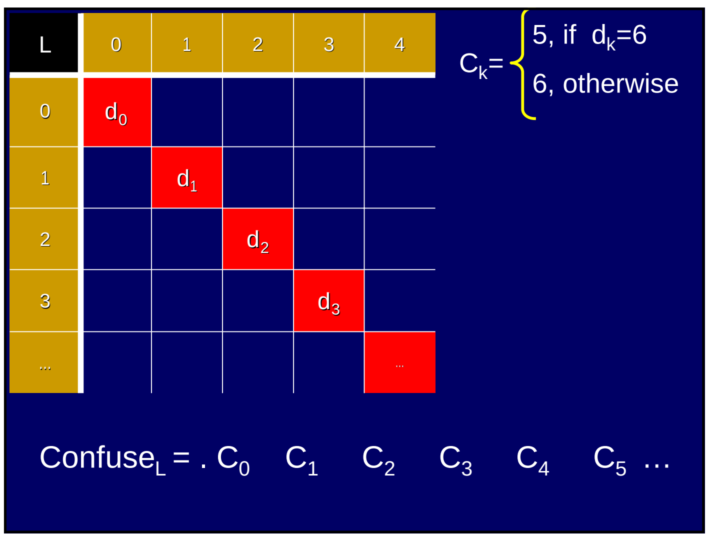
k: .345322214243555345221123235..

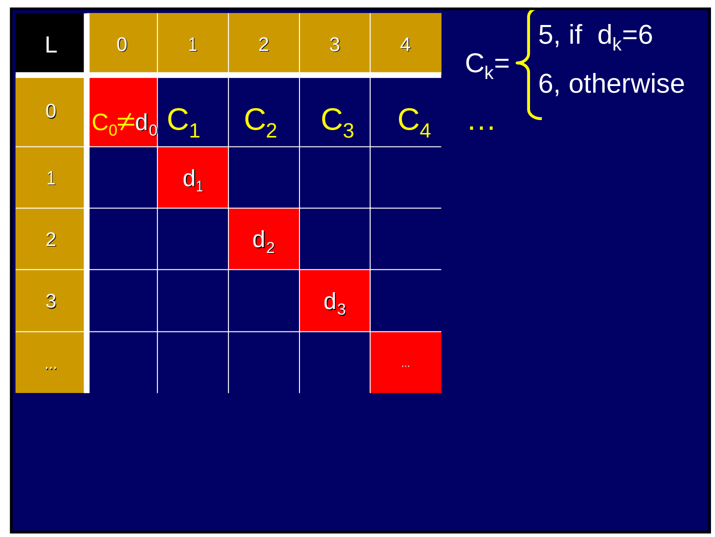
...

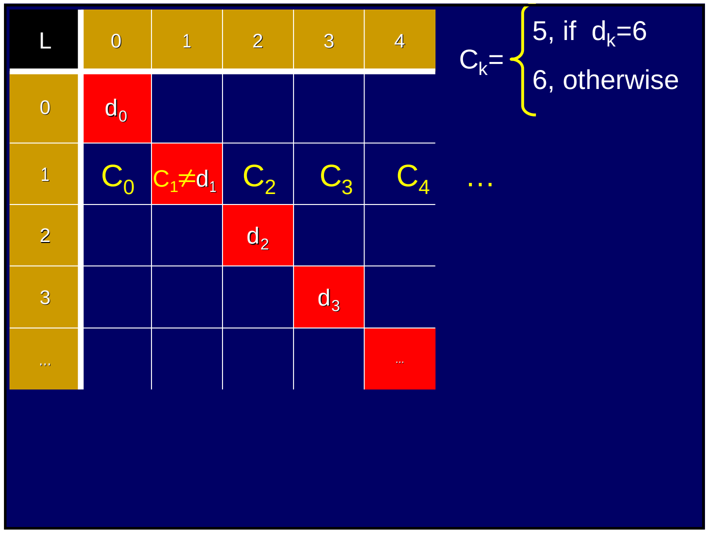
L	0	1	2	3	4	...
0						
1						
2						
3						
...						

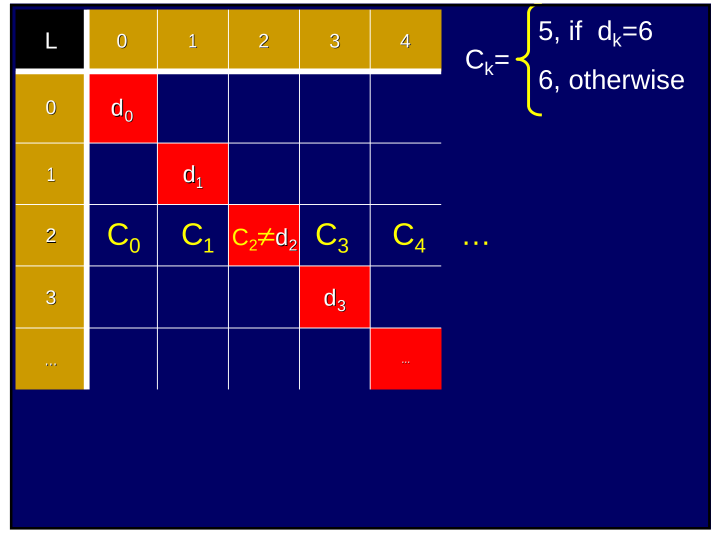

L
0
1
2
3
4
...
0
1
2
3
...
d_0
d_1
d_2
d_3
...

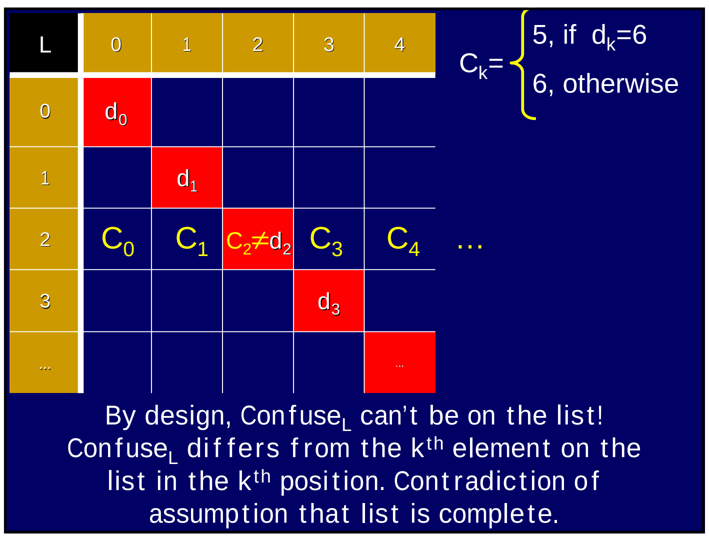

L
0
1
2
3
4
0
1
2
3
...
d_0
d_1
d_2
d_3
...
$Confuse_L = . C_0 \quad C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5 \ldots$


L
0
1
2
3
4
0
1
2
3
...
d_0
d_1
d_2
d_3
...
$C_k=$ 5, if $d_k=6$
6, otherwise
$Confuse_L = . C_0 \quad C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5 \ldots$

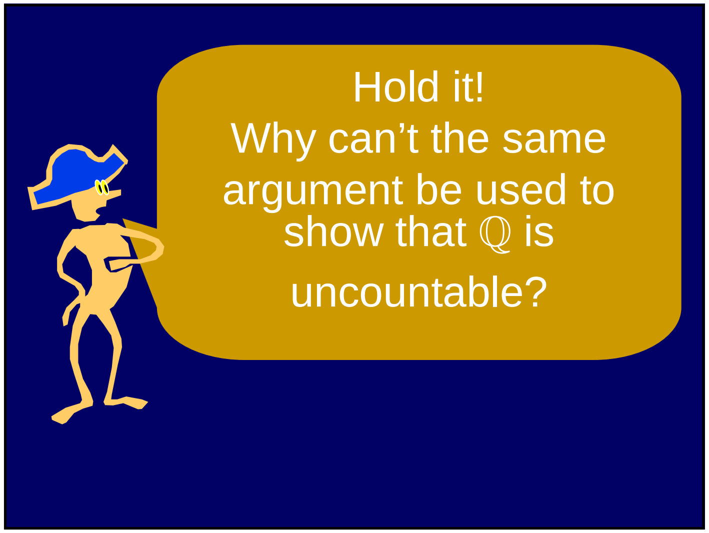

L
0
1
2
3
4
0
1
2
3
...
$C_0 \neq d_0$
C_1
C_2
C_3
C_4
…
d_1
d_2
d_3
...
$C_k=$ 5, if $d_k=6$
6, otherwise

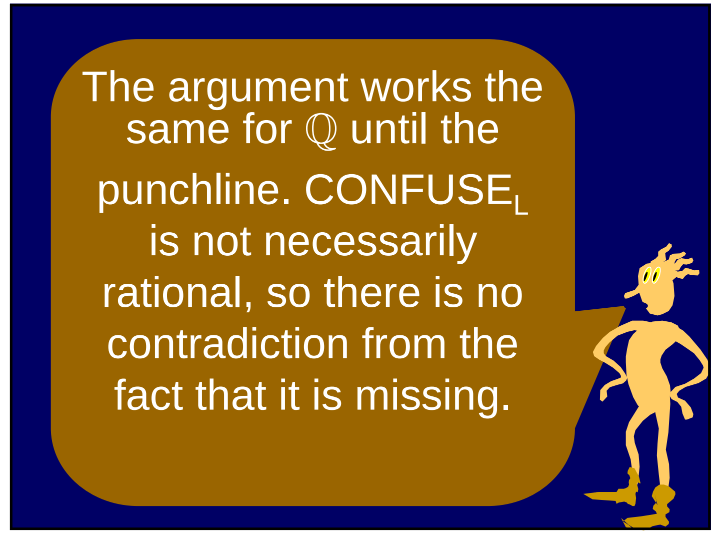

L
0 1 2 3 4
d_0
C_0 $C_1 \neq d_1$ C_2 C_3 C_4 …
d_2
d_3
…
$C_k = \begin{cases} 5, \text{ if } d_k=6 \\ 6, \text{ otherwise} \end{cases}$


L
0 1 2 3 4
d_0
d_1
C_0 C_1 $C_2 \neq d_2$ C_3 C_4 …
d_3
…
$C_k = \begin{cases} 5, \text{ if } d_k=6 \\ 6, \text{ otherwise} \end{cases}$


L
0 1 2 3 4
d_0
d_1
C_0 C_1 $C_2 \neq d_2$ C_3 C_4 …
d_3
…
$C_k = \begin{cases} 5, \text{ if } d_k=6 \\ 6, \text{ otherwise} \end{cases}$
By design, $Confuse_L$ can't be on the list! $Confuse_L$ differs from the k^{th} element on the list in the k^{th} position. Contradiction of assumption that list is complete.


The set of reals is uncountable!





Standard Notation

Σ = Any finite alphabet

Example: {a,b,c,d,e,…,z}

Σ* = All finite strings of symbols from Σ including the empty string ε

Theorem: Every infinite subset S of Σ* is countable

Proof: Sort S by first by length and then alphabetically. Map the first word to 0, the second to 1, and so on….

Stringing Symbols Together

Σ = The symbols on a standard keyboard

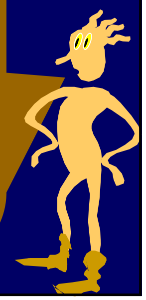
The set of all possible Java programs is a subset of Σ^*

The set of all possible finite pieces of English text is a subset of Σ^*

Thus:

The set of all possible Java programs is countable.

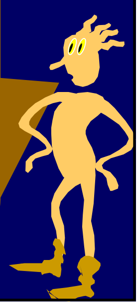
The set of all possible finite length pieces of English text is countable.

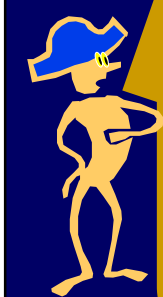


There are countably many Java program and uncountably many reals.

HENCE:

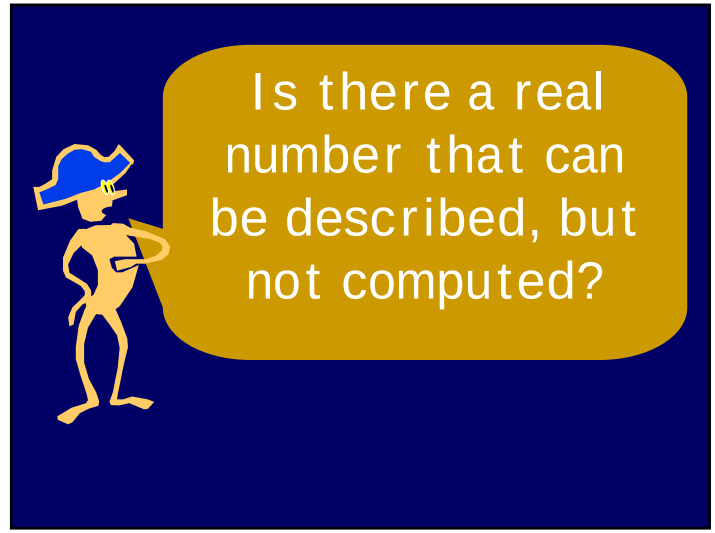
MOST REALS ARE NOT COMPUTABLE.





There are countably many descriptions and uncountably many reals.

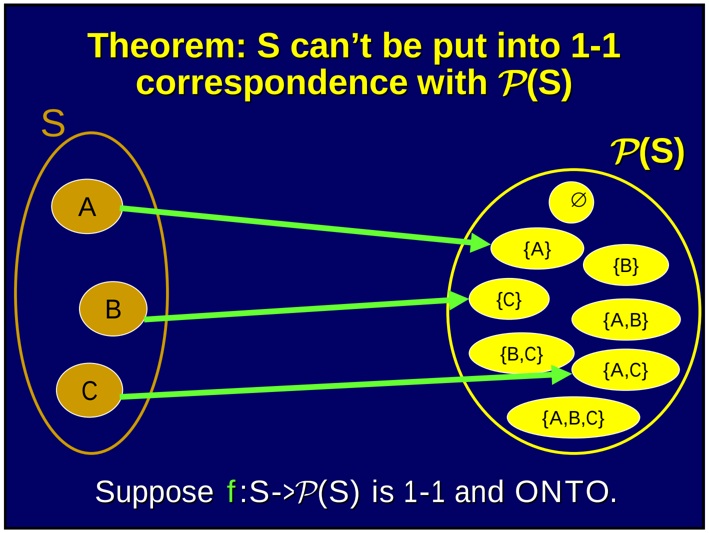
Hence:
MOST REAL NUMBERS ARE NOT DESCRIBEABLE!

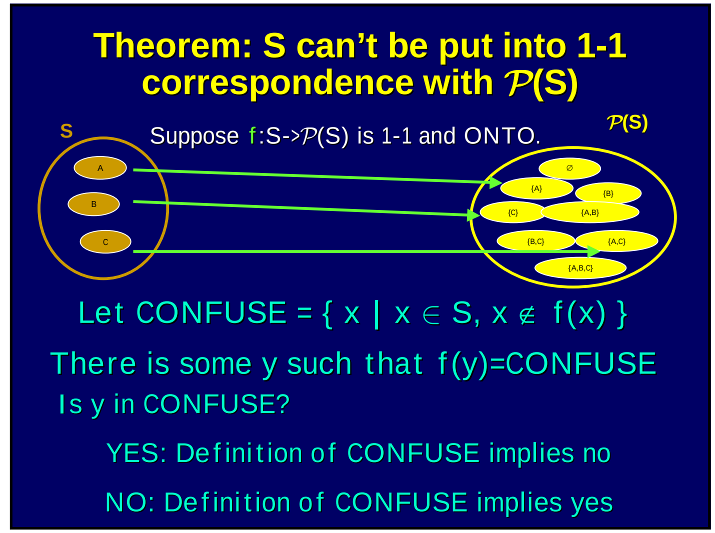


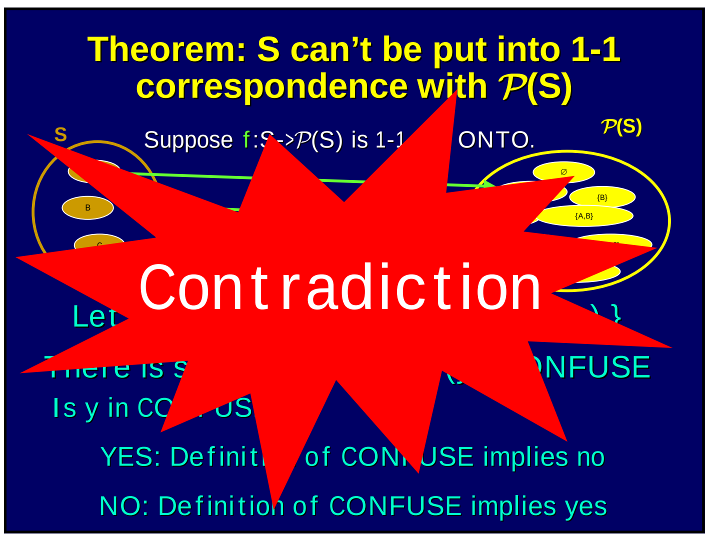
Power Set

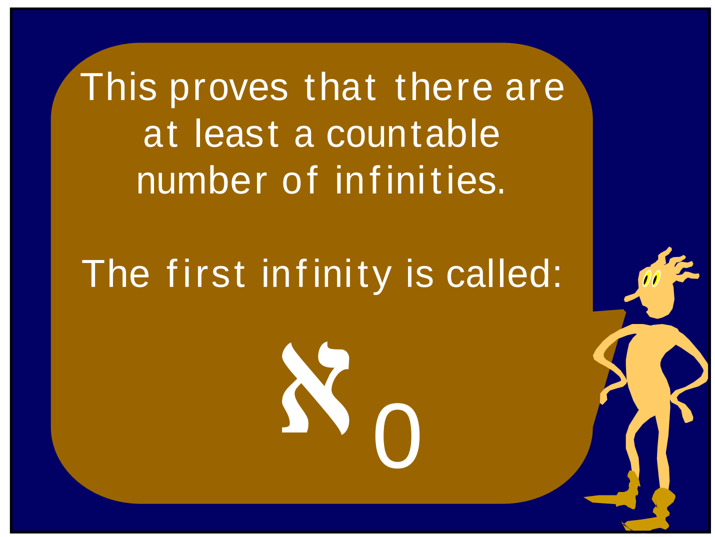
The power set of S is the set of all subsets of S. The power set is denoted $\mathcal{P}(S)$.

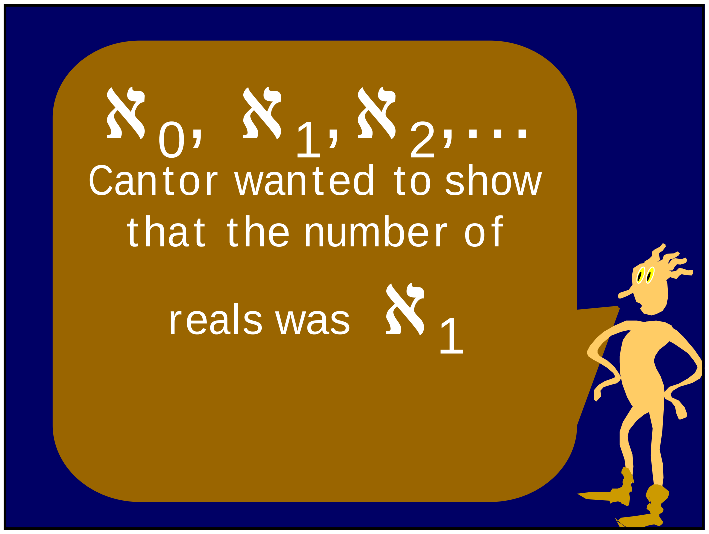
Proposition: If S is finite, the power set of S has cardinality $2^{|S|}$

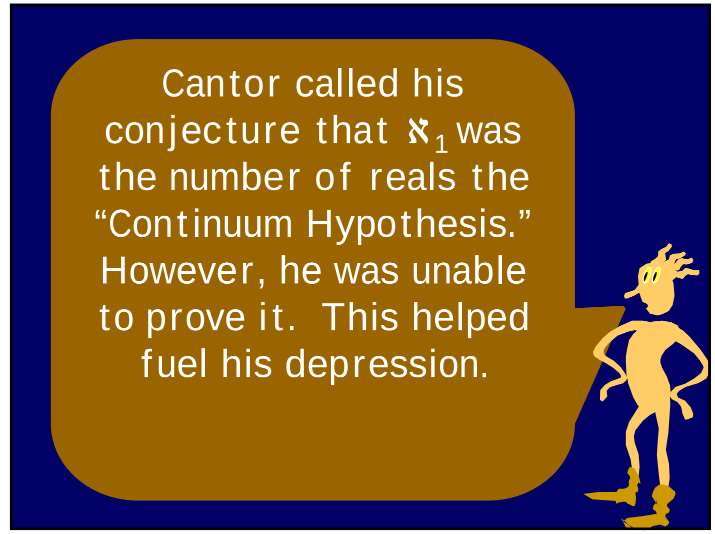

Theorem: S can't be put into 1-1 correspondence with 𝒫(S)
S
A
B
C
𝒫(S)
∅
{A}
{B}
{C}
{A,B}
{B,C}
{A,C}
{A,B,C}
Suppose f:S->𝒫(S) is 1-1 and ONTO.

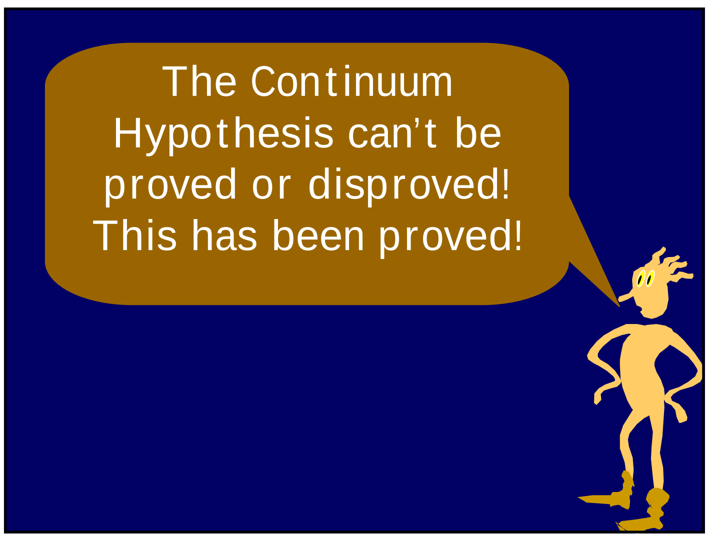

Theorem: S can't be put into 1-1 correspondence with 𝒫(S)
S
Suppose f:S->𝒫(S) is 1-1 and ONTO.
𝒫(S)
Let CONFUSE = { x | x ∈ S, x ∉ f(x) }
There is some y such that f(y)=CONFUSE
Is y in CONFUSE?
YES: Definition of CONFUSE implies no
NO: Definition of CONFUSE implies yes


Theorem: S can't be put into 1-1 correspondence with 𝒫(S)
Contradiction
NO: Definition of CONFUSE implies yes


This proves that there are at least a countable number of infinities.
The first infinity is called:
$\aleph_0$


$\aleph_0, \aleph_1, \aleph_2, \ldots$
Cantor wanted to show that the number of reals was $\aleph_1$


Cantor called his conjecture that $\aleph_1$ was the number of reals the “Continuum Hypothesis.” However, he was unable to prove it. This helped fuel his depression.


The Continuum Hypothesis can’t be proved or disproved! This has been proved!